Sheet 9

Ex 1 (i) $A = \begin{pmatrix} 3 & -1 & 0 \\ 3 & -1 & 0 \\ -1 & -1 & 2 \end{pmatrix}$

Characteristic polynomial = det$(\lambda I - A)$

$$p(\lambda) = \det(\lambda I - A)$$

$$\lambda I - A = \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 3 & -1 & 0 \\ 3 & -1 & 0 \\ -1 & -1 & 2 \end{pmatrix}$$

$$\lambda I - A = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} - \begin{pmatrix} 3 & -1 & 0 \\ 3 & -1 & 0 \\ -1 & -1 & 2 \end{pmatrix}$$

$$\lambda I - A = \begin{pmatrix} \lambda - 3 & 1 & 0 \\ -3 & \lambda + 1 & 0 \\ 1 & 1 & \lambda - 2 \end{pmatrix}$$

To compute the determinant we use co-factor expansion.

For co-factor expansion along the j^{th} column of matrix A:

$$\det A = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} M_{ij}$$

where a_{ij} is the entry in the i^{th} row, j^{th} column of matrix A.

M_{ij} is the determinant of the $(n-1)\times(n-1)$ matrix that results from A by removing the i^{th} row and j^{th} column of A, (where A is an $n\times n$ matrix.)

Expanding along the $j=3$ column.

$$p(\lambda) = (-1)^{3+1}(0)\begin{vmatrix} -3 & \lambda+1 \\ 1 & 1 \end{vmatrix}$$

$$+ (-1)^{3+2}(0)\begin{vmatrix} \lambda-3 & 1 \\ 1 & 1 \end{vmatrix}$$

$$+ (-1)^{3+3}(\lambda-2)\begin{vmatrix} \lambda-3 & 1 \\ -3 & \lambda+1 \end{vmatrix}$$

$$p(\lambda) = 0 + 0 + (\lambda-2)\begin{vmatrix} \lambda-3 & 1 \\ -3 & \lambda+1 \end{vmatrix}$$

$$p(\lambda) = (\lambda-2)\left[(\lambda-3)(\lambda+1) - (1)(-3)\right]$$

$$p(\lambda) = (\lambda-2)\left[\lambda^2 - 3\lambda + \lambda - 3 + 3\right]$$

$$p(\lambda) = (\lambda-2)\left[\lambda^2 - 2\lambda\right]$$

$$p(\lambda) = (\lambda-2)(\lambda)(\lambda-2)$$

$$p(\lambda) = (\lambda-2)^2(\lambda)$$

We find the eigenvalues of matrix A by setting $p(\lambda)=0$, and finding the roots.

$$\Rightarrow (\lambda-2)^2(\lambda) = 0$$

$$\Rightarrow \lambda-2=0, \quad \lambda=0$$

$$\Rightarrow \lambda=2, \quad \lambda=0$$

$\lambda=2$ has an algebraic multiplicity of 2.

The eigenvalues are:

$$\lambda_1 = 2, \quad \lambda_2 = 2, \quad \lambda_3 = 0.$$

For each eigenvalue $\lambda = \lambda_j$, we next look for eigenvectors as solutions of the linear system:

$$(\lambda I - A)\bar{x} = \bar{0}, \quad \text{where } \bar{x} \neq \bar{0}$$

$$\Rightarrow \begin{pmatrix} \lambda-3 & 1 & 0 \\ -3 & \lambda+1 & 0 \\ 1 & 1 & \lambda-2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

In the case of $\lambda = 2$

$$\begin{pmatrix} 2-3 & 1 & 0 \\ -3 & 2+1 & 0 \\ 1 & 1 & 2-2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -1 & 1 & 0 \\ -3 & 3 & 0 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Transform matrix to reduced row-echelon form

$$\Rightarrow \begin{pmatrix} -1 & 1 & 0 \\ -3 & 3 & 0 \\ 1 & 1 & 0 \end{pmatrix} \xrightarrow{\text{swap } \bar{r}_1 \,\&\, \bar{r}_2} \begin{pmatrix} 1 & 1 & 0 \\ -3 & 3 & 0 \\ -1 & 1 & 0 \end{pmatrix}$$

$$\xrightarrow{\bar{r}_1 + \bar{r}_3} \begin{pmatrix} 1 & 1 & 0 \\ -3 & 3 & 0 \\ 0 & 2 & 0 \end{pmatrix} \xrightarrow{\bar{r}_2 + 3\times\bar{r}_1} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 6 & 0 \\ 0 & 2 & 0 \end{pmatrix}$$

$$\xrightarrow{\bar{r}_2 \times \frac{1}{6}} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{pmatrix}$$

$$\xrightarrow{\bar{r}_3 - 2\times\bar{r}_2} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{\bar{r}_1 - \bar{r}_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Returning to our equation:

$$(\lambda I - A)\bar{x} = \bar{0}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} 1x_1 + 0x_2 + 0x_3 = 0 \\ 0x_1 + 1x_2 + 0x_3 = 0 \\ 0x_1 + 0x_2 + 0x_3 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} x_1 = 0 \\ x_2 = 0 \\ 0 = 0 \end{cases}$$

$$\boxed{\text{Basis vector for } \lambda = 2 \text{ eigenspace} \Rightarrow (0,0,1)}$$

$$\Rightarrow \bar{x} = t\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad t \in \mathbb{R}$$

Again considering our equation:

$(\lambda I - A)\bar{x} = \bar{0}$, now in the case where $\lambda = 0$

$$\Rightarrow \begin{pmatrix} -3 & 1 & 0 \\ -3 & 1 & 0 \\ 1 & 1 & -2 \end{pmatrix} \xrightarrow{\text{swap } \bar{r}_1 \,\&\, \bar{r}_3} \begin{pmatrix} 1 & 1 & -2 \\ -3 & 1 & 0 \\ -3 & 1 & 0 \end{pmatrix}$$

$$\xrightarrow{\bar{r}_3 - \bar{r}_2} \begin{pmatrix} 1 & 1 & -2 \\ -3 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{\bar{r}_2 + 3\times\bar{r}_1} \begin{pmatrix} 1 & 1 & -2 \\ 0 & 4 & -6 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{\bar{r}_2 \times \frac{1}{4}} \begin{pmatrix} 1 & 1 & -2 \\ 0 & 1 & -3/2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{\bar{r}_1 - \bar{r}_2} \begin{pmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -3/2 \\ 0 & 0 & 0 \end{pmatrix}$$

Returning to our equation

$$\Rightarrow \begin{pmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -3/2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} 1x_1 + 0x_2 - \frac{1}{2}x_3 = 0 \\ 0x_1 + 1x_2 - \frac{3}{2}x_3 = 0 \\ 0 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} x_1 = \frac{1}{2}x_3 \\ x_2 = \frac{3}{2}x_3 \\ x_3 = x_3 \end{cases}$$

x_3 is a free-variable, hence let $x_3 = \frac{1}{2}t$, where t is a free parameter.

$$\Rightarrow \begin{cases} x_1 = t \\ x_2 = 3t \\ x_3 = 2t \end{cases}, \quad t \in \mathbb{R}$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \quad t \in \mathbb{R}$$

$\Rightarrow$ basis vector for $\lambda = 0$ eigenspace $\Rightarrow (1, 3, 2)$

1 (ii)

$$A = \begin{pmatrix} 3 & -1 & 0 \\ 3 & -1 & 0 \\ 1 & -1 & 2 \end{pmatrix}$$

characteristic polynomial $p(\lambda) = \det(\lambda I - A)$

$$\lambda I - A = \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 3 & -1 & 0 \\ 3 & -1 & 0 \\ 1 & -1 & 2 \end{pmatrix}$$

$$\lambda I - A = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} - \begin{pmatrix} 3 & -1 & 0 \\ 3 & -1 & 0 \\ 1 & -1 & 2 \end{pmatrix}$$

$$\lambda I - A = \begin{pmatrix} \lambda-3 & 1 & 0 \\ -3 & \lambda+1 & 0 \\ -1 & 1 & \lambda-2 \end{pmatrix}$$

We must calculate the $\det(\lambda I - A)$.
We will use co-factor expansion along the $j=3$ column.

$$p(\lambda) = (-1)^{3+1}(0) \begin{vmatrix} -3 & \lambda+1 \\ -1 & 1 \end{vmatrix}$$

$$+ (-1)^{3+2}(0) \begin{vmatrix} \lambda-3 & 1 \\ -1 & 1 \end{vmatrix}$$

$$+ (-1)^{3+3}(\lambda-2) \begin{vmatrix} \lambda-3 & 1 \\ -3 & \lambda+1 \end{vmatrix}$$

$$p(\lambda) = 0 + 0 + (\lambda-2)\left[(\lambda-3)(\lambda+1) - (1)(-3)\right]$$

$$p(\lambda) = (\lambda-2)\left[\lambda^2 - 3\lambda + \lambda - 3 + 3\right]$$

$$p(\lambda) = (\lambda-2)\left[\lambda^2 - 2\lambda\right]$$

$$p(\lambda) = (\lambda-2)(\lambda)(\lambda-2)$$

$$p(\lambda) = (\lambda-2)^2(\lambda)$$

we find the eigenvalues of matrix A by setting $p(\lambda) = 0$, and finding the roots.

$$\Rightarrow \quad (\lambda-2)^2(\lambda) = 0$$

$$\Rightarrow \quad \lambda - 2 = 0 \quad , \quad \lambda = 0$$

$$\Rightarrow \quad \lambda = 2 \quad , \quad \lambda = 0$$

$\lambda = 2$ has an algebraic multiplicity of 2.

The eigenvalues are :

$\lambda_1 = 2$, $\lambda_2 = 2$, $\lambda_3 = 0$

For each eigenvalue $\lambda = \lambda_j$, we next look for eigenvectors as solutions of the linear system:

$$(\lambda I - A)\bar{x} = \bar{0}, \qquad \text{where } \bar{x} \neq \bar{0}.$$

$$\Rightarrow \begin{pmatrix} \lambda-3 & 1 & 0 \\ -3 & \lambda+1 & 0 \\ -1 & 1 & \lambda-2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

In the case of $\lambda = 2$

$$\begin{pmatrix} 2-3 & 1 & 0 \\ -3 & 2+1 & 0 \\ -1 & 1 & 2-2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -1 & 1 & 0 \\ -3 & 3 & 0 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Transform matrix to reduced-row echelon form.

$$\Rightarrow \begin{pmatrix} -1 & 1 & 0 \\ -3 & 3 & 0 \\ -1 & 1 & 0 \end{pmatrix} \xrightarrow{\bar{r}_3 - \bar{r}_1} \begin{pmatrix} -1 & 1 & 0 \\ -3 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{\bar{r}_2 - 3\times\bar{r}_1} \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{\bar{r}_1 \times -1} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Returning to our equation:

$$(\lambda I - A)\bar{x} = \bar{0}$$

$$\Rightarrow \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} x_1 - x_2 + 0x_3 = 0 \\ 0x_1 + 0x_2 + 0x_3 = 0 \\ 0x_1 + 0x_2 + 0x_3 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} x_1 = x_2 \\ 0 = 0 \\ 0 = 0 \end{cases}$$

$\Rightarrow$ x_2 and x_3 are free variables.

Let $x_2 = t$,
$x_3 = s$

where t, s are free parameters.

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Basis vectors for $\lambda = 2$ eigenspace: $(1,1,0)$, $(0,0,1)$.

Now for $\lambda = 0$ eigenvalue

$$\Rightarrow \begin{pmatrix} 0-3 & 1 & 0 \\ -3 & 0+1 & 0 \\ -1 & 1 & 0-2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Transform matrix to reduced-row echelon form.

$$\Rightarrow \begin{pmatrix} -3 & 1 & 0 \\ -3 & 1 & 0 \\ -1 & 1 & -2 \end{pmatrix} \xrightarrow{\bar{r}_2 - \bar{r}_1} \begin{pmatrix} -3 & 1 & 0 \\ 0 & 0 & 0 \\ -1 & 1 & -2 \end{pmatrix}$$

$$\xrightarrow{\text{swap } \bar{r}_1 \,\&\, \bar{r}_3} \begin{pmatrix} -1 & 1 & -2 \\ 0 & 0 & 0 \\ -3 & 1 & 0 \end{pmatrix} \xrightarrow{\bar{r}_1 \times -1} \begin{pmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \\ -3 & 1 & 0 \end{pmatrix}$$

$$\xrightarrow{\text{swap } \bar{r}_2 \,\&\, \bar{r}_3} \begin{pmatrix} 1 & -1 & 2 \\ -3 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{\bar{r}_2 + 3\times\bar{r}_1} \begin{pmatrix} 1 & -1 & 2 \\ 0 & -2 & 6 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{\bar{r}_2 \times \frac{-1}{2}} \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{\bar{r}_1 + \bar{r}_2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{pmatrix}$$

Returning to our equation:

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} 1x_1 + 0x_2 - 1x_3 = 0 \\ 0x_1 + 1x_2 - 3x_3 = 0 \\ 0x_1 + 0x_2 + 0x_3 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} x_1 = x_3 \\ x_2 = 3x_3 \\ 0 = 0 \end{cases}$$

x_3 is a free variable.

Let $x_3 = t$, where t is a free parameter.

$$\Rightarrow \begin{cases} x_1 = t \\ x_2 = 3t \\ x_3 = t \end{cases}$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}, \quad t \in \mathbb{R}$$

~~Th~~ A Basis vector for $\lambda = 0$ eigenspace : $(1, 3, 1)$.

Ex 1(iii) $A = \begin{pmatrix} 3 & -1 & 0 \\ 3 & -1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$

Characteristic polynomial $p(\lambda) = \det(\lambda I - A)$

$$(\lambda I - A) \Rightarrow \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 3 & -1 & 0 \\ 3 & -1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$$

$$(\lambda I - A) \Rightarrow \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} - \begin{pmatrix} 3 & -1 & 0 \\ 3 & -1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$$

$$(\lambda I - A) \Rightarrow \begin{pmatrix} \lambda-3 & 1 & 0 \\ -3 & \lambda+1 & 0 \\ -1 & 1 & \lambda-1 \end{pmatrix}$$

We must calculate the $\det(\lambda I - A)$.
We will use co-factor expansion along the $j=3$ column.

$$p(\lambda) = (-1)^{3+1}(0) \begin{vmatrix} \cancel{\lambda}-3 & \lambda+1 \\ -1 & 1 \end{vmatrix}$$

$$+ (-1)^{3+2}(0) \begin{vmatrix} \lambda-3 & 1 \\ -1 & 1 \end{vmatrix}$$

$$+ (-1)^{3+3}(\lambda-1) \begin{vmatrix} \lambda-3 & 1 \\ -3 & \lambda+1 \end{vmatrix}$$

$$p(\lambda) = 0 + 0 + (\lambda - 1)\left[(\lambda-3)(\lambda+1) - (1)(-3)\right]$$

$$p(\lambda) = (\lambda - 1)\left[\lambda^2 - 3\lambda + \lambda - 3 + 3\right]$$

$$p(\lambda) = (\lambda - 1)\left[\lambda^2 - 2\lambda\right]$$

$$p(\lambda) = (\lambda - 1)(\lambda)(\lambda - 2)$$

we find the eigenvalues of matrix A by setting $p(\lambda) = 0$, and finding the roots

$$\Rightarrow (\lambda - 1)(\lambda)(\lambda - 2) = 0$$

$$\Rightarrow \lambda = 1, \quad \lambda = 0, \quad \lambda = 2$$

The eigenvalues are

$$\lambda_1 = 1, \quad \lambda_2 = 0, \quad \lambda_3 = 2.$$

For each eigenvalue $\lambda = \lambda_j$, we next look for eigenvectors as solutions of the linear system:

$$\Rightarrow \begin{pmatrix} \lambda - 3 & 1 & 0 \\ -3 & \lambda + 1 & 0 \\ -1 & 1 & \lambda - 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

where $\bar{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

For each $\lambda = \lambda_j$, we transform the matrix to reduced-row echelon form.

For $\lambda_1 = 1$,

$$\Rightarrow \begin{pmatrix} 1-3 & 1 & 0 \\ -3 & 1+1 & 0 \\ -1 & 1 & 1-1 \end{pmatrix} = \begin{pmatrix} -2 & 1 & 0 \\ -3 & 2 & 0 \\ -1 & 1 & 0 \end{pmatrix}$$

$$\xrightarrow{\text{swap } \bar{r}_3 \text{ \& } \bar{r}_1} \begin{pmatrix} -1 & 1 & 0 \\ -3 & 2 & 0 \\ -2 & 1 & 0 \end{pmatrix} \xrightarrow{\bar{r}_1 \times -1} \begin{pmatrix} 1 & -1 & 0 \\ -3 & 2 & 0 \\ -2 & 1 & 0 \end{pmatrix}$$

$$\xrightarrow{\bar{r}_2 + 3\times\bar{r}_1} \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 0 \\ -2 & 1 & 0 \end{pmatrix} \xrightarrow{\bar{r}_2 \times -1} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ -2 & 1 & 0 \end{pmatrix}$$

$$\xrightarrow{\bar{r}_3 + 2\times\bar{r}_1} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix} \xrightarrow{\bar{r}_3 + \bar{r}_2} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{\bar{r}_1 + \bar{r}_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} 1x_1 + 0x_2 + 0x_3 = 0 \\ 0x_1 + 1x_2 + 0x_3 = 0 \\ 0x_1 + 0x_2 + 0x_3 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} x_1 = 0 \\ x_2 = 0 \\ 0 = 0 \end{cases}$$

$\Rightarrow x_3$ is a free variable.

let $x_3 = t$, where t is a free parameter

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad , \quad t \in \mathbb{R}$$

A basis vector for $\lambda = 1$ eigenspace $\Rightarrow (0, 0, 1)$.

For $\lambda_2 = 0$

$$\Rightarrow \begin{pmatrix} -3 & 1 & 0 \\ -3 & 1 & 0 \\ -1 & 1 & -1 \end{pmatrix} \xrightarrow{\text{swap } \bar{r}_1 \,\&\, \bar{r}_3} \begin{pmatrix} -1 & 1 & -1 \\ -3 & 1 & 0 \\ -3 & 1 & 0 \end{pmatrix}$$

$$\xrightarrow{\bar{r}_1 \times -1} \begin{pmatrix} 1 & -1 & 1 \\ -3 & 1 & 0 \\ -3 & 1 & 0 \end{pmatrix} \xrightarrow{\bar{r}_3 - \bar{r}_2} \begin{pmatrix} 1 & -1 & 1 \\ -3 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{\bar{r}_2 + 3 \times \bar{r}_1} \begin{pmatrix} 1 & -1 & 1 \\ 0 & -2 & 3 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{\bar{r}_2 \times -\frac{1}{2}} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -3/2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{\bar{r}_1 + \bar{r}_2} \begin{pmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -3/2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -3/2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} 1x_1 + 0x_2 - \frac{1}{2}x_3 = 0 \\ 0x_1 + 1x_2 - \frac{3}{2}x_3 = 0 \\ 0x_1 + 0x_2 + 0x_3 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} x_1 = \frac{1}{2}x_3 \\ x_2 = \frac{3}{2}x_3 \\ 0 = 0 \end{cases}$$

$\Rightarrow x_3$ is a free variable.

Let $x_3 = 2t$, where t is a free parameter.

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \quad t \in \mathbb{R}$$

A basis for $\lambda = 0$ eigenspace

$\Rightarrow (1,3,2)$.

For $\lambda_3 = 2$

$$\Rightarrow \begin{pmatrix} 2-3 & 1 & 0 \\ -3 & 2+1 & 0 \\ -1 & 1 & 2-1 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 \\ -3 & 3 & 0 \\ -1 & 1 & 1 \end{pmatrix}$$

$$\xrightarrow{\bar{r}_1 \times -1} \begin{pmatrix} 1 & -1 & 0 \\ -3 & 3 & 0 \\ -1 & 1 & 1 \end{pmatrix} \xrightarrow{\bar{r}_2 + 3\times\bar{r}_1} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ -1 & 1 & 1 \end{pmatrix}$$

$$\xrightarrow{\text{swap } \bar{r}_2 \,\&\, \bar{r}_3} \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \sim$$

$$\xrightarrow{\bar{r}_1 + \bar{r}_2} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} x_1 - x_2 + 0x_3 = 0 \\ 0x_1 + 0x_2 + 1x_3 = 0 \\ 0x_1 + 0x_2 + 0x_3 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} x_1 = x_2 \\ x_3 = 0 \\ 0 = 0 \end{cases}$$

x_2 is a free variable, let $x_2 = t$ where t is a free parameter.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} , \quad t \in \mathbb{R}$$

A basis for $\lambda = 2$ eigenspace

$\Rightarrow (1, 1, 0)$

Ex 2 To find the matrices P and D we make use of the following theorem:

For any basis of $\mathbb{R}^n$ consisting of eigenvectors $\bar{v}_1, \dots, \bar{v}_n$ of an $n \times n$ matrix A, the matrix P formed by $\bar{v}_j$'s as columns is invertible and diagonalizes A, i.e. conjugates into the diagonal matrix,

$$P^{-1} A P = D = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$$

where λ_j is the eigenvalue of $\bar{v}_j$ for each j.

For matrix A in 1(iii) we have 3 distinct eigenvalues and we can thus make use of the following theorem:

"A set of eigenvectors with distinct eigenvalues is linearly independent."

Hence, the eigenvectors $\bar{e}_1 = (0,0,1)$, $\bar{e}_2 = (1,3,2)$ and $\bar{e}_3 = (1,1,0)$ form a basis in $\mathbb{R}^3$.

Thus for matrix A in 1(iii) we have

$$P = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 3 & 1 \\ 1 & 2 & 0 \end{pmatrix} \text{ and } D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

For matrix A in 1(ii) there are 2 distinct eigenvalues $\lambda = 2$ and $\lambda = 0$.

A basis for the $\lambda = 0$ eigenspace is the vector $(1, 3, 1)$.

Hence the vector $(1, 3, 1)$ will be linearly independent with respect to any eigenvector of $\lambda = 2$.

A basis for $\lambda = 2$ eigenspace contains two vectors $(1, 1, 0)$ and $(0, 0, 1)$. Because these two vectors form a basis, they must be definition be linearly independent of each other.

Hence the set of eigenvectors: $(1,3,1)$, $(1,1,0)$ & $(0,0,1)$ are linearly independent and the number of vectors in the set equals the dimension of $\mathbb{R}^3$.

Hence the set of eigenvectors:
$\{(1,3,1), (1,1,0), (0,0,1)\}$ form a basis for $\mathbb{R}^3$.

$$\Rightarrow P = \begin{pmatrix} 1 & 1 & 0 \\ 3 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

For the matrix A in 1 (i) there are two distinct eigenvalues $\lambda = 0$ and $\lambda = 2$.

A basis for the ~~e~~ $\lambda = 0$ eigenspace contains only 1 vector.
A basis for the $\lambda = 2$ eigenspace contains only 1 vector.

Hence it is not possible to find three ~~d~~ linearly independent eigenvectors.

For this matrix, there is no set of eigenvectors that forms a basis for $\mathbb{R}^3$.

Hence it is not possible to find matrices P and D.

Ex 3 Taking $A = \begin{pmatrix} 3 & -1 & 0 \\ 3 & -1 & 0 \\ 1 & -1 & 2 \end{pmatrix}$

we need to find a general solution to the system of ordinary differential equations

$$\begin{pmatrix} y_1' \\ y_2' \\ y_3' \end{pmatrix} = A \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

we use a change of co-ordinates

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = P \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

differentiate

$$\begin{pmatrix} y_1' \\ y_2' \\ y_3' \end{pmatrix} = P \begin{pmatrix} u_1' \\ u_2' \\ u_3' \end{pmatrix}$$

$$\Rightarrow P \begin{pmatrix} u_1' \\ u_2' \\ u_3' \end{pmatrix} = A P \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} u_1' \\ u_2' \\ u_3' \end{pmatrix} = P^{-1} A P \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} u_1' \\ u_2' \\ u_3' \end{pmatrix} = D \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

As shown in 2(ii) in this case:

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} u_1' \\ u_2' \\ u_3' \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} u_1' \\ u_2' \\ u_3' \end{pmatrix} = \begin{pmatrix} 0u_1 + 0u_2 + 0u_3 \\ 0u_1 + 2u_2 + 0u_3 \\ 0u_1 + 0u_2 + 2u_3 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} u_1' \\ u_2' \\ u_3' \end{pmatrix} = \begin{pmatrix} 0 \\ 2u_2 \\ 2u_3 \end{pmatrix}$$

$$\Rightarrow \begin{cases} u_1' = 0u_1 \\ u_2' = 2u_2 \\ u_3' = 2u_3 \end{cases}$$

Given a first order, linear differential equation of the form:

$$y' = ay$$

where $y = f(x)$ is an unknown differential function, $y' = f'(x)$ is a derivative and a is a constant.

The general solution is given as:

$$y = Ce^{ax},$$

where C is an arbitary constant.

General solutions of our equations are given as

$$\begin{cases} u_1 = C_1 e^{0x} \\ u_2 = C_2 e^{2x} \\ u_3 = C_3 e^{2x} \end{cases}$$

$$\Rightarrow \begin{cases} u_1 = C_1 \\ u_2 = C_2 e^{2x} \\ u_3 = C_3 e^{2x} \end{cases}$$

$$\& \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = P \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 3 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 e^{2x} \\ C_3 e^{2x} \end{pmatrix}$$

$$\Rightarrow \begin{cases} y_1 = C_1 + C_2 e^{2x} + 0 \cdot C_3 e^{2x} \\ y_2 = 3C_1 + C_2 e^{2x} + 0 \cdot C_3 e^{2x} \\ y_3 = C_1 + 0 \cdot C_2 e^{2x} + C_3 e^{2x} \end{cases}$$

$$\Rightarrow \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} + C_2 e^{2x} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + C_3 e^{2x} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

3(ii) Initial value condition: $y(0) = (1, 0, -1)$

$$\& \quad \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 3 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 e^{2x} \\ C_3 e^{2x} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 3 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 e^{2(0)} \\ C_3 e^{2(0)} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 3 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix}$$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 3 & 1 & 0 & 0 \\ 1 & 0 & 1 & -1 \end{array}\right)$$

$$\xrightarrow{\bar{r}_3 - \bar{r}_1} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 3 & 1 & 0 & 0 \\ 0 & -1 & 1 & -2 \end{array}\right)$$

$$\xrightarrow{\bar{r}_2 - 3\times\bar{r}_1} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & -2 & 0 & -3 \\ 0 & -1 & 1 & -2 \end{array}\right)$$

$$\xrightarrow{\bar{r}_2 \times -\frac{1}{2}} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 3/2 \\ 0 & -1 & 1 & -2 \end{array}\right)$$

$$\xrightarrow{\bar{r}_3 + \bar{r}_2} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 3/2 \\ 0 & 0 & 1 & -1/2 \end{array}\right)$$

$$\xrightarrow{\bar{r}_1 - \bar{r}_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1/2 \\ 0 & 1 & 0 & 3/2 \\ 0 & 0 & 1 & -1/2 \end{array}\right)$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} -1/2 \\ 3/2 \\ -1/2 \end{pmatrix}$$

$$\Rightarrow \begin{cases} c_1 + 0c_2 + 0c_3 = -1/2 \\ 0c_1 + 1c_2 + 0c_3 = 3/2 \\ 0c_1 + 0c_2 + 1c_3 = -1/2 \end{cases}$$

$$\Rightarrow \begin{cases} c_1 = -1/2 \\ c_2 = 3/2 \\ c_3 = -1/2 \end{cases}$$

$$\Rightarrow \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} + \frac{3}{2}e^{2x}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \frac{1}{2}e^{2x}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\text{or } \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 3 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} -1/2 \\ 3/2 \cdot e^{2x} \\ -1/2\, e^{2x} \end{pmatrix}$$

Ex 4 To compute A^{50} we make use of the following formula:

$$A^n = P D^n P^{-1}$$

Taking $A = \begin{pmatrix} 3 & -1 & 0 \\ 3 & -1 & 0 \\ 1 & -1 & 2 \end{pmatrix}$

we know that for this matrix A the matrices P and D are given as:

$$P = \begin{pmatrix} 1 & 1 & 0 \\ 3 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \text{ \& } D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

we need to find P^{-1}.
Make use of the following theorem:
For $n \times n$ matrices P, R, I, where I is the identity matrix, assume

$$(P \mid I) \rightsquigarrow (I \mid R)$$

means row equivalent

Then $R = P^{-1}$.

$$\Rightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 3 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{\bar{r}_3 - \bar{r}_1} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 3 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{\bar{r}_2 - 3\times\bar{r}_1} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -2 & 0 & -3 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{\bar{r}_2 \times -\frac{1}{2}} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 3/2 & -1/2 & 0 \\ 0 & -1 & 1 & -1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{\bar{r}_3 + \bar{r}_2} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 3/2 & -1/2 & 0 \\ 0 & 0 & 1 & 1/2 & -1/2 & 1 \end{array}\right)$$

$$\xrightarrow{\bar{r}_1 - \bar{r}_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/2 & 1/2 & 0 \\ 0 & 1 & 0 & 3/2 & -1/2 & 0 \\ 0 & 0 & 1 & 1/2 & -1/2 & 1 \end{array}\right)$$

$$\Rightarrow \quad P^{-1} = \begin{pmatrix} -1/2 & 1/2 & 0 \\ 3/2 & -1/2 & 0 \\ 1/2 & -1/2 & 1 \end{pmatrix}$$

$$A^{50} = \begin{pmatrix} 1 & 1 & 0 \\ 3 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2^{50} & 0 \\ 0 & 0 & 2^{50} \end{pmatrix} \times \begin{pmatrix} -1/2 & 1/2 & 0 \\ 3/2 & -1/2 & 0 \\ 1/2 & -1/2 & 1 \end{pmatrix}$$

$$A^{50} = \begin{pmatrix} 1 & 1 & 0 \\ 3 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \times$$

$$\begin{pmatrix} 0\left(\frac{-1}{2}\right)+0\left(\frac{3}{2}\right)+0\left(\frac{1}{2}\right) & 0\left(\frac{1}{2}\right)+0\left(-\frac{1}{2}\right)+0\left(-\frac{1}{2}\right) & 0(0)+0(0)+0(1) \\ 0\left(-\frac{1}{2}\right)+2^{50}\left(\frac{3}{2}\right)+0\left(\frac{1}{2}\right) & 0\left(\frac{1}{2}\right)+2^{50}\left(-\frac{1}{2}\right)+0\left(-\frac{1}{2}\right) & 0(0)+2^{50}(0)+0(1) \\ 0\left(-\frac{1}{2}\right)+0\left(\frac{3}{2}\right)+2^{50}\left(\frac{1}{2}\right) & 0\left(\frac{1}{2}\right)+0\left(-\frac{1}{2}\right)+2^{50}\left(-\frac{1}{2}\right) & 0(0)+0(0)+2^{50}(1) \end{pmatrix}$$

$$A^{50} = \begin{pmatrix} 1 & 1 & 0 \\ 3 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 0 & 0 & 0 \\ 3\cdot 2^{49} & -2^{49} & 0 \\ 2^{49} & -2^{49} & 2^{50} \end{pmatrix}$$

$$A^{50} = \begin{pmatrix} 1(0)+1(3\cdot 2^{49})+0(2^{49}) & 1(0)+1(-2^{49})+0(-2^{49}) & 1(0)+1(0)+0(2^{50}) \\ 3(0)+1(3\cdot 2^{49})+0(2^{49}) & 3(0)+1(-2^{49})+0(-2^{49}) & 3(0)+1(0)+0(2^{50}) \\ 1(0)+0(3\cdot 2^{49})+1(2^{49}) & 1(0)+0(-2^{49})+1(-2^{49}) & 1(0)+0(0)+1(2^{50}) \end{pmatrix}$$

$$A^{50} = \begin{pmatrix} 3\cdot 2^{49} & -2^{49} & 0 \\ 3\cdot 2^{49} & -2^{49} & 0 \\ 2^{49} & -2^{49} & 2^{50} \end{pmatrix}$$

Taking
$$A = \begin{pmatrix} 3 & -1 & 0 \\ 3 & -1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$$

Again we compute A^{50} using the formula:
$$A^n = P D^n P^{-1}$$

we know for this matrix A, the matrices P and D are given as:

$$P = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 3 & 1 \\ 1 & 2 & 0 \end{pmatrix} \qquad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Now, we need to find P^{-1} using the following theorem:

For $n \times n$ matrices P, R, I, where I is the identity matrix, assume

$$(P \mid I) \xrightarrow[\text{(row equivalent)}]{} (I \mid R)$$

Then $R = P^{-1}$

$$\Rightarrow \left(\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{\text{Swap } \bar{r}_1 \,\&\, \bar{r}_3} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 0 & 1 \\ 0 & 3 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{\bar{r}_2 \times \frac{1}{3}} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1/3 & 0 & 1/3 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{\bar{r}_3 - \bar{r}_2} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1/3 & 0 & 1/3 & 0 \\ 0 & 0 & 2/3 & 1 & -1/3 & 0 \end{array}\right)$$

$$\xrightarrow{\bar{r}_3 \times 3/2} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1/3 & 0 & 1/3 & 0 \\ 0 & 0 & 1 & 3/2 & -1/2 & 0 \end{array}\right)$$

$$\xrightarrow{\bar{r}_2 - \frac{1}{3} \times \bar{r}_3} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1/2 & 1/2 & 0 \\ 0 & 0 & 1 & 3/2 & -1/2 & 0 \end{array}\right)$$

$$\xrightarrow{\bar{r}_1 - 2 \times \bar{r}_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & -1/2 & 1/2 & 0 \\ 0 & 0 & 1 & 3/2 & -1/2 & 0 \end{array}\right)$$

$$P^{-1} = \begin{pmatrix} 1 & -1 & 1 \\ -1/2 & 1/2 & 0 \\ 3/2 & -1/2 & 0 \end{pmatrix}$$

$$A^{50} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 3 & 1 \\ 1 & 2 & 0 \end{pmatrix} \times \begin{pmatrix} 1^{50} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2^{50} \end{pmatrix} \times \begin{pmatrix} 1 & -1 & 1 \\ -1/2 & 1/2 & 0 \\ 3/2 & -1/2 & 0 \end{pmatrix}$$

$$\Rightarrow A^{50} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 3 & 1 \\ 1 & 2 & 0 \end{pmatrix} \times$$

$$\begin{pmatrix} 1(1)+0(-\frac{1}{2})+0(\frac{3}{2}) & 1(-1)+0(\frac{1}{2})+0(-\frac{1}{2}) & 1(1)+0(0)+0(0) \\ 0(1)+0(-\frac{1}{2})+0(\frac{3}{2}) & 0(-1)+0(\frac{1}{2})+0(-\frac{1}{2}) & 0(1)+0(0)+0(0) \\ 0(1)+0(-\frac{1}{2})+2^{50}(\frac{3}{2}) & 0(-1)+0(\frac{1}{2})+2^{50}(-\frac{1}{2}) & 0(1)+0(0)+0(2^{50}) \end{pmatrix}$$

$$A^{50} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 3 & 1 \\ 1 & 2 & 0 \end{pmatrix} \times \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 3\cdot 2^{49} & -2^{49} & 0 \end{pmatrix}$$

$$A^{50} = \begin{pmatrix} 0(1)+1(0)+1(3\cdot 2^{49}) & 0(-1)+1(0)+1(-2^{49}) & 0(1)+1(0)+1(0) \\ 0(1)+3(0)+1(3\cdot 2^{49}) & 0(-1)+3(0)+1(-2^{49}) & 0(1)+3(0)+1(0) \\ 1(1)+2(0)+0(3\cdot 2^{49}) & 1(-1)+2(0)+0(-2^{49}) & 1(1)+2(0)+0(0) \end{pmatrix}$$

$$A^{50} = \begin{pmatrix} 3\cdot 2^{49} & -2^{49} & 0 \\ 3\cdot 2^{49} & -2^{49} & 0 \\ 1 & -1 & 1 \end{pmatrix}$$